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Flow along a horizontal plate near a free surface

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The problem of flow along a horizontal semi-infinite flat plate moving in its own plane through a viscous liquid just below the free surface is considered. The method of matched asymptotic expansions is used to analyse the interaction between the free surface and the boundary layer formed on the plate. It is found that, due to viscosity, small-amplitude gravity waves on the free surface can be formed. The formulae for the resistance of the plate containing the free-surface effect and for the lift, appearing as a new phenomenon, are derived.

1. Introduction

While investigating the broad subject of water waves, we often encounter the intriguing problem of the interaction between waves and the boundary layer formed on a body in a viscous stream. Although this wave–viscosity interaction has not, perhaps, been in the forefront of recent theoretical investigations, it still has great practical importance. For instance, in order to determine the resistance of a ship, naval architects face the problem in everyday practice. In fact, they avoid it by following Froude's assumption (Froude 1877) that the resistance consists of two superimposed parts – frictional and residual resistance, the main part of the latter being the resistance due to waves. The methods for determining ship resistance have developed from Froude's time (see e.g. Lewis 1988) but his main idea, the independent analysis of waves and viscosity, still has great practical value. The question is, to what extent is that assumption correct.

Many authors tried to improve the predicted ship resistance by incorporating the neglected wave–viscosity interaction into the problem: Havelock (1935), Inui (1957), Wigley (1963), Wu (1963), Maruo (1976), Mori (1979) and Stern (1986), to mention just a few attempts in the long history, but the problem, with its own ever-perplexing nature, still lies open.

In that context, it is rather curious that a much more restricted problem, but one that seems important and instructive for understanding the phenomenon, has not been (as far as the author knows) theoretically solved. The problem in question is the flow along a semi-infinite horizontal plate in a stream of viscous liquid with a free surface (figure 1). Its importance, in connection with the wave–boundary layer interaction, can be seen immediately. If, following Froude, we try to analyse waves and viscosity separately, and at first neglect viscosity, we directly conclude that such a plate would cause no disturbances in the oncoming horizontal stream. There would be no waves on the free surface, and certainly no wave resistance. On the other hand, if the liquid is viscous but the free surface is not present, it would be the well-known boundary-layer problem – for laminar flow, the classical Blasius problem (Blasius 1908). We conclude that in the problem under consideration (viscous liquid with a free surface), all the disturbances (waves), if any, formed on the horizontal free surface would be caused by the effect of viscosity. Also, all the deviations from the Blasius flow are due to the presence of the free surface, and (as the classical solution is well known) would be easily traced. So the problem, in spite of the simplifications (a flat plate may seem a poor approximation of a ship), contains the main characteristics of wave–boundary layer interaction and, in addition, presents them in a simple and obvious way. The fact that it is a natural generalization of the classical Blasius problem, and that is suitable for analytical treatment, gives more significance to the problem studied.

2. Basic equations

The problem presented in figure 1 will now be defined more precisely. A flat horizontal semi-infinite plate that moves with constant velocity u_0 through still viscous liquid at a depth h under its free surface is considered. All the disturbances of the flow are caused entirely by the moving plate, and they are supposed to be laminar. In the coordinate system x, y moving with the plate the flow is, under these assumptions, steady and two-dimensional, governed by the Navier–Stokes equations and the equation of continuity:

$$uu_x + vu_y = -\frac{1}{F}p'_x + \frac{1}{R}(u_{xx} + u_{yy}),$$

$$uv_x + vv_y = -\frac{1}{F}p'_y + \frac{1}{R}(v_{xx} + v_{yy}),$$

$$u_x + v_y = 0.$$

The equations are presented in non-dimensional form, scales for velocity, length and pressure being u_0, h and $p_0 = \rho g h$ respectively, where ρ is the liquid density, and g acceleration due to gravity. In those equations u, v represents the velocity components, and F and R Froude and Reynolds numbers, defined as

$$F = u_0^2/gh, \quad R = u_0 h/\nu$$

where ν is the kinematic viscosity. The gravitational force has been cancelled by the hydrostatic pressure, so p' denotes the pressure disturbance.

The following boundary conditions must be satisfied: undisturbed flow far upstream

$$u = 1, \quad p' = v = \zeta = 0 \quad \text{for} \quad x \to -\infty$$

(ζ is the free surface disturbance); a no-slip condition on the plate

$$u = v = 0 \quad \text{for} \quad y = 0, \quad x > 0;$$

and the kinematic and dynamic conditions on the free surface

$$\left.\begin{aligned} v &= u\zeta_x \\ (p' - \zeta)\,\zeta_x + \frac{F}{R}(u_y + v_x - 2u_x\,\zeta_x) &= 0 \\ p' - \zeta + \frac{F}{R}[(u_y + v_x)\,\zeta_x - 2v_y] &= 0 \end{aligned}\right\} \quad \text{for} \quad y = 1 + \zeta(x).$$

As we are interested in the gravity wave–viscosity interaction, the capillary effects in the dynamic condition on the free surface are neglected.

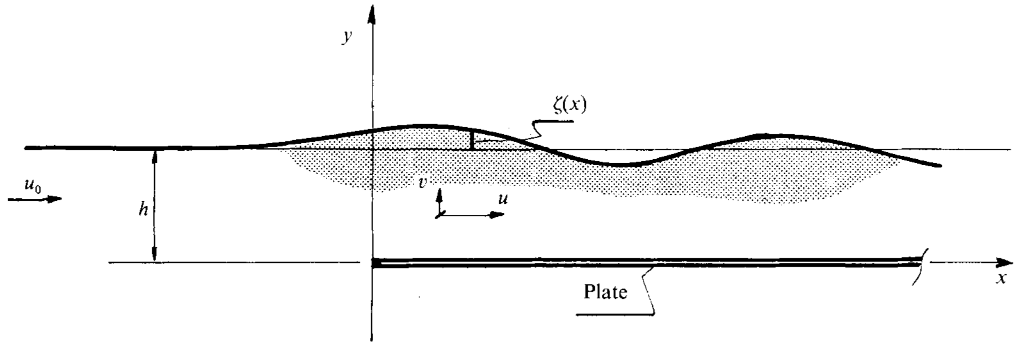

FIGURE 1. Configuration of the problem.

If the stream function ψ is introduced in the usual way:

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x,$$

these equations and the boundary conditions become

$$\left.\begin{aligned} \psi_y\psi_{xy} - \psi_x\psi_{yy} &= -\frac{1}{F}p'_x + \frac{1}{R}(\psi_{yxx} + \psi_{yyy}), \\ \psi_y\psi_{xx} - \psi_x\psi_{xy} &= \frac{1}{F}p'_y + \frac{1}{R}(\psi_{xxx} + \psi_{yyx}), \end{aligned}\right\} \tag{1a}$$

$$\psi = y \quad \text{for} \quad x \to -\infty, \tag{1b}$$

$$\psi_y = 0, \quad \psi_x = 0 \quad \text{for} \quad y = 0, \quad x > 0, \tag{1c}$$

$$\left.\begin{aligned} \psi_x &= -\zeta_x\psi_y \\ (p' - \zeta)\zeta_x + \frac{F}{R}(\psi_{yy} - \psi_{xx} - 2\zeta_x\psi_{xy}) &= 0 \\ p' - \zeta + \frac{F}{R}[(\psi_{yy} - \psi_{xx})\zeta_x - 2\psi_{xy}] &= 0 \end{aligned}\right\} \quad \text{for} \quad y = 1 + \zeta(x). \tag{1d}$$

The problem stated above possesses nonlinear and viscous terms in the equations of motion and in the conditions on the free surface, so we are not able to solve it in all its complexity. We limit our investigation to the case of weak interaction between the plate and the surface, and for that reason make two more assumptions concerning parameters R and F. We suppose $R \gg 1$ and note that such an assumption leads not only to a thin boundary layer on the plate, but as $R^{\frac{1}{2}}$ is proportional to h/δ (where δ is the boundary-layer thickness at the distance h from the leading edge), the assumption implies that the surface is far outside the boundary layer as long as $x \ll R$ is satisfied. The other assumption we make is $F/R \ll 1$. This makes the viscous terms in the free-surface conditions small enough to be neglected in the procedure that follows, leaving the outer flow (to the order considered) entirely inviscid in nature.

Our aim was to make possible the use of perturbation methods, and we achieved that by restricting R and F, but unfortunately left the problem of strong interaction $R = O(1)$ (where the viscosity would affect the whole layer between the plate and the surface) beyond the scope of this investigation. However, a wide variety of flows is still covered, as demonstrated by noting that for a plate, for example, 5 cm under the surface of water, the conditions $R \gg 1$, $F/R \ll 1$ would be well satisfied for the range of velocities $u_0 = 2$ mm/s–2 m/s.

Although the use of the depth h as a scale, instead of an arbitrary length, does not restrict the analysis that follows, and is even beneficial because the result will depend only on two non-dimensional parameters (R and F), such a scaling should be carefully noted. The fact that h will not enter the results explicitly must not lead to the conclusion that the flow does not depend on it. The depth will be hidden in parameters R, F and non-dimensional variables throughout the paper. So, if an explicit variation of the flow with h is needed, either a dimensional form of the results has to be recovered, or the scale for length changed to an arbitrary one, and that could be (at every stage of the analysis) easily done. Of course, from such transformed results, the limit $h \to \infty$ will recover the classical flow with no free surface.

After this necessary discussion on the parameters and scales involved, we are ready to start solving the above problem. Knowing the appropriate solutions with no free surface (see Van Dyke 1975), all the unknown functions are expanded in terms of the parameter $1/R^{\frac{1}{2}}$, separately for the outer flow and for the inner flow (the boundary layer). For the outer flow

$$\left.\begin{aligned} \psi(x,y) &= \psi^{(0)}(x,y)+\frac{1}{R^{\frac{1}{2}}}\psi^{(1)}(x,y)+\frac{1}{R}\psi^{(2)}(x,y)+\dots, \\ p'(x,y) &= \frac{1}{R^{\frac{1}{2}}}p^{(1)}(x,y)+\frac{1}{R}p^{(2)}(x,y)+\dots, \\ \zeta(x) &= \frac{1}{R^{\frac{1}{2}}}\zeta^{(1)}(x)+\frac{1}{R}\zeta^{(2)}(x)+\dots; \end{aligned}\right\} \tag{2}$$

and for the inner flow

$$\left.\begin{aligned} \psi &= \psi(x,Y) = \frac{1}{R^{\frac{1}{2}}}\Psi^{(0)}(x,Y)+\frac{1}{R}\Psi^{(1)}(x,Y)+\dots, \\ p' &= p'(x,Y) = \frac{1}{R^{\frac{1}{2}}}P^{(1)}(x,Y)+\frac{1}{R}P^{(2)}(x,Y)+\dots, \end{aligned}\right\} \tag{3}$$

where the inner coordinate $Y = yR^{\frac{1}{2}}$ is introduced. Now, to solve this problem, the well-known method of matched asymptotic expansions is used. Briefly, the expansions (2) and (3) are substituted into the equations and boundary conditions (1), terms of the same order are equated and the expansions that represent the same physical quantities are matched in the region where the outer and the inner flows overlap. In that way, instead of the problem defined by the equations and boundary conditions (1), the series of simpler problems for the unknown functions

$$\psi^{(N)}, \quad p^{(N+1)}, \quad \zeta^{(N+1)}, \quad \Psi^{(N)}, \quad P^{(N+1)}, \quad N = 0,1,2,\dots,$$

are obtained, called the first-, second-, etc. order approximations of the problem.

3. The first-order approximation

As the first-order approximation of the outer flow, the following equation for the stream function is obtained:

$$\nabla^2\psi^{(0)} = 0,$$

with the boundary conditions

$$\psi^{(0)} = y \quad \text{for} \quad x \to -\infty,$$
$$\psi^{(0)}_x = 0 \quad \text{for} \quad y = 1,$$

where the conditions on the plate (1 *c*) have been dropped, and the conditions on the free surface (1 *d*) have been expanded about the undisturbed state $y = 1$. The solution of the problem is obvious:

$$\psi^{(0)} = y,$$

and represents the undisturbed horizontal stream.

The first-order approximation of the inner flow is

$$\Psi^{(0)}_{YYY} - \Psi^{(0)}_Y \Psi^{(0)}_{xY} + \Psi^{(0)}_x \Psi^{(0)}_{YY} = 0, \tag{4a}$$
$$P^{(1)}_Y = 0, \tag{4b}$$

with the boundary conditions on the plate

$$\Psi^{(0)}_x = \Psi^{(0)}_Y = 0 \quad \text{for} \quad Y = 0. \tag{5}$$

The third boundary condition for the equation (4*a*) is obtained by matching with the outer flow. It follows that

$$\Psi^{(0)}_Y \to 1 \quad \text{for} \quad Y \to \infty. \tag{6}$$

In equations (4) and boundary conditions (5) and (6) we recognize the classical Blasius boundary-layer problem which, by introducing the new variables

$$\eta = \frac{Y}{(2x)^{\frac{1}{2}}}, \quad f(\eta) = \pm \frac{\Psi^{(0)}}{(2x)^{\frac{1}{2}}},$$

reduces to the Blasius differential equation

$$f''' + ff'' = 0,$$

with the boundary conditions:

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

Without going into details, just the solutions of this well-known problem for small and large η are quoted here, as they are essential for the analysis that follows:

$$f(\eta) \approx \tfrac{1}{2}\alpha\eta^2, \quad \alpha = 0.4696... \quad \text{for} \quad \eta \ll 1,$$
$$f(\eta) \sim \eta - \beta, \quad \beta = 1.21678... \quad \text{for} \quad \eta \gg 1.$$

The solution obtained – the undisturbed stream for the outer flow (as if the liquid were inviscid), and Blasius flow for the boundary layer (as if the free surface were not present) – should be understood as a proof that the usual separate analysis of viscosity and waves is, as a first approximation of the problem, correct. So, the free surface–boundary layer interaction has to be looked for in the higher-order approximations.

4. The second-order approximation – outer flow

Substitution of the outer expansions (2) into the equations and boundary conditions (1) (the conditions on the free surface (1 *d*) having already been expanded about the undisturbed state and the conditions on the plate (1 *c*) dropped) yields the equations

$$\nabla^2 \psi^{(1)} = 0, \tag{7a}$$
$$p^{(1)} = -F\psi^{(1)}_y, \tag{7b}$$

with the boundary conditions

$$\psi^{(1)} \to 0 \quad \text{for} \quad x \to -\infty, \tag{8a}$$

$$\left.\begin{aligned} \psi_x^{(1)} - F\psi_{xy}^{(1)} &= 0 \\ \zeta^{(1)} &= -F\psi_y^{(1)} \end{aligned}\right\} \quad \text{for} \quad y = 1 \tag{8b, c}$$

Matching with the first-order approximation of the inner flow gives

$$\psi_x^{(1)} = \mp\frac{\beta}{(2x)^{\frac{1}{2}}} \quad \text{for} \quad x > 0, \quad y = 0,$$

where the upper and the lower sign on the right-hand side stand for the upper and the lower side of the plate, respectively.

There is a familiar physical interpretation of the problem stated above – it is the inviscid flow past a thin parabolic cylinder of nose radius β^2/R. But, in contrast to the second-order approximation of the classical Blasius problem, the free surface is involved. From the point of view of the water wave theory, we are dealing here with the waves caused by an obstacle in the steady stream. Methods for solving this type of water wave problem, for different shapes of obstacle, are well known (Wehausen & Laitone 1960). The particular one of the obstacle being a thin parabolic cylinder seems not to have been published yet. So, it will be analysed here briefly, while the details are given in Hofman (1986). We use the old ideas (see Sretenski 1977), and assume that the flow disturbance consists of two parts, the disturbance caused by the parabolic cylinder in the unbounded stream, and the correction associated with the presence of the free surface:

$$\psi^{(1)} = \psi^{(p)} + \psi^{(w)}.$$

The first, well-known part of the disturbance,

$$\psi^{(p)} = -\beta \operatorname{sgn} y\, [(x^2+y^2)^{\frac{1}{2}} + x]^{\frac{1}{2}},$$

has to be expressed in the form of a Fourier integral. To do that, we start from the Fourier transform:

$$\mathrm{FT}\,[x^{\frac{1}{2}} H(x)] = \tfrac{1}{2}!\,|k|^{\frac{3}{2}} \exp(-\tfrac{3}{4}\pi \mathrm{i} \operatorname{sgn} k)$$

(Lighthill 1958), where $H(x)$ is Heaviside's function and i is the imaginary unit. The boundary value follows:

$$\psi^{(p)}(x,0) = -\beta(2x)^{\frac{1}{2}} H(x) \operatorname{sgn} y = \frac{\beta(1+\mathrm{i})}{4\pi^{\frac{1}{2}}} \operatorname{sgn} y \int_{-\infty}^{\infty} \frac{\operatorname{sgn} k}{k^{\frac{3}{2}}} \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k,$$

and from it we deduce that

$$\psi^{(p)}(x,y) = \frac{\beta(1+\mathrm{i})}{4\pi^{\frac{1}{2}}} \operatorname{sgn} y \int_{-\infty}^{\infty} \frac{\operatorname{sgn} k\, \mathrm{e}^{-|ky|}}{k^{\frac{3}{2}}} \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k. \tag{9}$$

The second part of the disturbance is also expressed in the form of an integral:

$$\psi^{(w)} = \int_{-\infty}^{\infty} \Phi(y,k)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x, \tag{10}$$

where the unknown function $\Phi(y,k)$ is introduced. To obtain that elementary

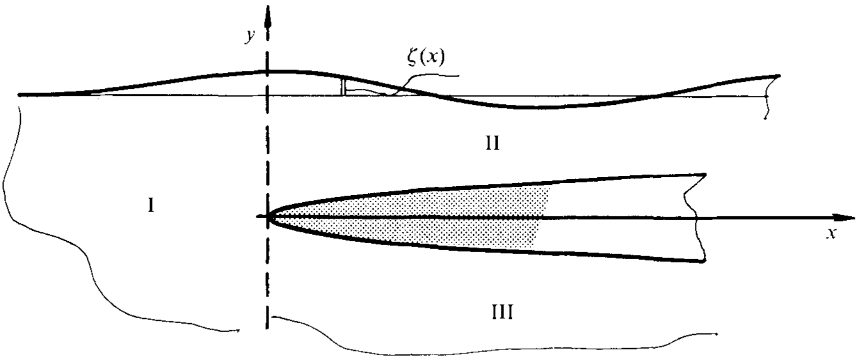

FIGURE 2. Characteristic regions of the outer flow.

solution we have to divide the flow field into the three regions (figure 2). *Region I* ($x < 0$) – infinitely deep flow with a free surface, where we suppose

$$\Phi(y,k) = A(k)\exp[|k|(y-1)]; \tag{11}$$

region II ($x > 0$, $y > 0$) – flow of slowly varying depth with a free surface, where we suppose

$$\Phi(y,k) = A(k)\sinh ky; \tag{12}$$

and *region III* ($x > 0$, $y < 0$) – infinitely deep flow with no free surface. It is elementary to prove that in this last region

$$\Phi(y,k) = 0, \quad \psi^{(w)} = 0$$

is valid. Consequently, in that region, the classical result for the flow over a plate in an unbounded stream follows (Van Dyke 1975):

$$u^{(1)} = 0, \tag{13}$$

that part of the flow field being not affected by the free surface.

The procedure that follows for the other two regions is straightforward. Integrals (9) and (10) are substituted into the free-surface condition (8*b*) (using the assumptions (11) and (12) respectively) and the appropriate values of $A(k)$ are found. We get

$$\left.\begin{aligned} A(k) &= -\frac{\beta(1+\mathrm{i})\,\mathrm{e}^{-|k|}}{4\pi^{\frac{1}{2}}}\frac{\operatorname{sgn}k+Fk}{k^{\frac{3}{2}}\,1-F(k-2\mathrm{i}\epsilon)\operatorname{sgn}k} \quad \text{in region I,}\\ A(k) &= -\frac{\beta(1+\mathrm{i})\,\mathrm{e}^{-|k|}}{4\pi^{\frac{1}{2}}}\frac{\operatorname{sgn}k+Fk}{k^{\frac{3}{2}}\,\sinh k-F(k-2\mathrm{i}\epsilon)\cosh k} \quad \text{in region II,} \end{aligned}\right\} \tag{14}$$

where, in order to satisfy the radiation condition, a small positive parameter ϵ was introduced (see Lighthill 1978). It is sufficient for our purpose just to find the streamwise component of velocity $u^{(1)} = \psi_y^{(1)}$, and then, by use of the interface condition (8*c*), obtain the free-surface disturbance. So, from (9), (10), (11), (12) and (14) it follows that

$$\left.\begin{aligned} u^{(1)} &= -\frac{\beta(1+\mathrm{i})}{4\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{(1-F|k|)\,\mathrm{e}^{-|ky|}+(1+F|k|)\,\mathrm{e}^{|k|(y-2)}}{k^{\frac{1}{2}}[1-F(k-2\mathrm{i}\epsilon)\operatorname{sgn}k]}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \quad \text{in region I,}\\ u^{(1)} &= -\frac{\beta(1+\mathrm{i})}{4\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{\cosh[k(1-y)]-Fk\sinh[k(1-y)]}{k^{\frac{1}{2}}[\sinh k-F(k-2\mathrm{i}\epsilon)\cosh k]}\operatorname{sgn}k\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \quad \text{in region II.} \end{aligned}\right\} \tag{15}$$

To evaluate (15) we divide the integrals into regions $k < 0$ and $k > 0$ and use $m = |k|$ as the variable of integration. We obtain

$$u^{(1)} = -\frac{\beta(1+\mathrm{i})}{4\pi^{\frac{1}{2}}}(J_1 - \mathrm{i}J_2), \tag{16}$$

where

$$J_{1,2} = \int_0^\infty \frac{(1-Fm)\,\mathrm{e}^{-m|y|} + (1+Fm)\,\mathrm{e}^{m(y-2)}}{m^{\frac{1}{2}}[1-F(m \mp 2\mathrm{i}\epsilon)]}\,\mathrm{e}^{\pm \mathrm{i}kx}\,\mathrm{d}m \quad \text{in region I,}$$

$$J_{1,2} = \int_0^\infty \frac{\cosh[m(1-y)] - Fm\sinh[m(1-y)]}{m^{\frac{1}{2}}[\sinh m - F(m \mp 2\mathrm{i}\epsilon)\cosh m]}\,\mathrm{e}^{\pm \mathrm{i}kx}\,\mathrm{d}m \quad \text{in region II.}$$

In region I the integrand has a pole $m = 1/F$ which is, because of the small positive parameter ϵ, just above the real axes for J_1, and just below the real axes for J_2. The path of integration has to be rotated into negative and positive imaginary axes, respectively. The pole does not contribute, so after we neglect ϵ (which was introduced just to shift the pole off the path of integration) and use the relation (16), we get

$$\left.\begin{aligned} u^{(1)} &= -\frac{\beta}{4\pi^{\frac{1}{2}}}\int_0^\infty \frac{\sigma(m,y)}{m^{\frac{1}{2}}}\,\mathrm{e}^{mx}\,\mathrm{d}m,\\ \sigma(m,y) &= \frac{(1+F^2m^2)\cos(my) + (1-F^2m^2)\cos[m(y-2)] - 2Fm\sin[m(y-2)]}{1+F^2m^2}\\ &\qquad \text{in region I.}\end{aligned}\right\} \tag{17}$$

In region II, the integrand has an infinite number of imaginary poles $m = \pm \mathrm{i}k_n$, $n = 1, 2, 3, \ldots$, where the k_n are real positive solutions of the equation

$$Fk = \tan k,$$

and, for $F < 1$ only, two poles $m = \pm k_0$, where k_0 is a real positive solution of the equation

$$Fk = \tanh k.$$

Now, the path of integration has to be rotated into positive and negative imaginary axes respectively. One of the real poles $m = k_0$ (because of small positive ϵ) is inside the path of integration for both integrals J_1 and J_2. The imaginary poles $m = \mathrm{i}k_n$ contribute to J_1 and $m = -\mathrm{i}k_n$ to J_2. By the use of relation (16) the contributions of the real pole to J_1 and J_2 are added, while the contributions of the imaginary poles are cancelled. So (after neglecting ϵ) we obtain

$$\left.\begin{aligned} u^{(1)} &= b_w \sin(k_0 x + \tfrac{1}{4}\pi) - \frac{\beta}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \frac{\sigma(m,y)}{m^{\frac{1}{2}}}\,\mathrm{e}^{-mx}\,\mathrm{d}m \quad \text{for} \quad F < 1,\\ u^{(1)} &= -\frac{\beta}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \frac{\sigma(m,y)}{m^{\frac{1}{2}}}\,\mathrm{e}^{-mx}\,\mathrm{d}m \quad \text{for} \quad F > 1,\\ \sigma(m,y) &= \frac{\cos[m(1-y)] + Fm\sin[m(1-y)]}{\sin m - Fm\cos m} \quad \text{in region II,}\end{aligned}\right\} \tag{18}$$

where for the first time the wave amplitude appears:

$$b_w(F,y) = \frac{\beta(2\pi)^{\frac{1}{2}}\cosh(k_0 y)}{k_0^{\frac{1}{2}}[1 - F/(1-F^2k_0^2)]}. \tag{19}$$

Finally, to find the integrals in (17) and (18) in the form of asymptotic expansions, the appropriate functions $\sigma(m, y)$ are expanded in powers of m and the resultant series integrated term by term.

In region I (ahead of the cylinder), such a procedure leads to solutions for $u^{(1)}$ and $\zeta^{(1)}$ in the form of the expansions:

$$u^{(1)} = \sum_{N=1}^{\infty} \frac{a_N(F,y)}{|x|^{2N-\frac{3}{2}}}, \qquad \zeta^{(1)} = \sum_{N=1}^{\infty} \frac{A_N(F)}{|x|^{2N-\frac{3}{2}}}, \tag{20}$$

where the coefficients a_N and A_N are

$$a_1 = -\beta\sqrt{2},$$

$$a_2 = \frac{3\beta\sqrt{2}}{16}[y^2 - 4y(1-F) - 8F + 4F^2], \quad \ldots,$$

$$A_N = -Fa_N(F, 1),$$

which, obviously, represent just a local disturbance of the flow, disappearing far ahead of the leading edge.

In region II (over the cylinder) the situation becomes much more interesting because of the pole $m = k_0$ that exists for $F < 1$ only, and is physically connected to waves of length $\lambda_0 = 2\pi/k_0$. So, in that region two different flow regimes can exist: subcritical ($F < 1$), and supercritical ($F > 1$). The solutions for these regimes are

$$\left.\begin{aligned} u^{(1)} &= b_w(F,y)\sin(k_0 x + \tfrac{1}{4}\pi) + \frac{\beta(2\pi)^{\frac{1}{2}}}{1-F} + \sum_{N=1}^{\infty} \frac{b_N(F,y)}{x^{2N-\frac{1}{2}}} \\ \zeta^{(1)} &= B_w(F)\sin(k_0 x + \tfrac{1}{4}\pi) - \frac{\beta F(2x)^{\frac{1}{2}}}{1-F} + \sum_{N=1}^{\infty} \frac{B_N(F)}{x^{2N-\frac{1}{2}}} \end{aligned}\right\} \quad \text{for} \quad F < 1, \tag{21}$$

and

$$\left.\begin{aligned} u^{(1)} &= \frac{\beta(2x)^{\frac{1}{2}}}{1-F} + \sum_{N=1}^{\infty} \frac{b_N(F,y)}{x^{2N-\frac{1}{2}}} \\ \zeta^{(1)} &= -\frac{\beta F(2x)^{\frac{1}{2}}}{1-F} + \sum_{N=1}^{\infty} \frac{B_N(F)}{x^{2N-\frac{1}{2}}} \end{aligned}\right\} \quad \text{for} \quad F > 1. \tag{22}$$

The wavenumber in (21) is defined by $Fk_0 = \tanh k_0$, the amplitude b_w is given by (19), the amplitude of surface waves is $B_w = -Fb_w(F, 1)$, and the coefficients b_N and B_N are

$$\left.\begin{aligned} b_1 &= \frac{\beta}{12\sqrt{2}(1-F)^2}\{(3F-1) + 3(1-y)(1-F)[(1-y)-2F]\}, \quad \ldots, \\ B_N &= -Fb_N(F, 1). \end{aligned}\right\} \tag{23}$$

The flow disturbance over the cylinder consists of three separate parts: the local disturbance (the last terms in (21) and (22)), the parabolic disturbance (the $x^{\frac{1}{2}}$ term), and, in the case of subcritical flow only, the wave disturbance of the stream. On adding the presence or absence of waves, there is also another change in the flow as the Froude number passes one. For $F < 1$ the parabolic part of the free-surface disturbance is a depression, while for $F > 1$ it is an elevation that follows the parabolic cylinder. Both differences between subcritical and supercritical flow are familiar from water wave theory. The stationary waves cannot exist on the water stream for $u_0 > (gh)^{\frac{1}{2}}$ (see e.g. Lighthill 1978), while a hump on the bottom produces a depression on the stream surface for $u_0 < (gh)^{\frac{1}{2}}$, and a rise for $u_0 > (gh)^{\frac{1}{2}}$, that have exactly the forms we obtained (see Paterson 1983).

All three parts of the disturbance (local, parabolic and wave) grow unboundedly in the case of the transcritical flow ($F \approx 1$), where the method used is unable to give an acceptable solution. The reason seems clear. The present method relies on the assumption that a small viscosity (high R) causes a small change in the oncoming stream. However, as the Froude number approaches its critical value, because of the strong interaction between viscosity and the free surface, the presence of even a slight viscosity causes abrupt changes in the flow. So that basic assumption is no longer satisfied.

There is another singularity in the solution obtained. The solution is not bounded near the leading edge of the cylinder. The explanation is similar to the one just mentioned, but (in contrast to the case of transcritical flow) this difficulty will be overcome. The unknown flow for small x will not prevent us from accomplishing the main goal: to obtain the resistance of the plate including the leading edge.

Further discussion on the parabolic part of the flow disturbance seems desirable. It is obtained, formally, that for $F < 1$ the parabolic depression of the free surface would intersect the cylinder far downstream. Of course, the solutions (in accordance with the method used) are valid only for small $\zeta(x)$, that is for $x \ll R$. Furthermore, there arises the problem of the turbulent boundary layer far downstream. Keeping in mind that the far-downstream flow is beyond the problem treated in this paper, one still cannot avoid the question: what would happen to the free surface in the case of a solid parabolic cylinder (not the boundary layer) far down the inviscid stream? We will try to answer it briefly. The local Froude number

$$F^* = u_0^2/gh^*(x)$$

(where $h^*(x)$ is local depth) grows with x for the subcritical flow, so every subcritical flow becomes (for large x) transcritical, and finally, in the far downstream region, supercritical. There, in accordance with the solution obtained (and the physical reasoning) the free surface follows the growth of the parabolic cylinder.

Leaving this far-downstream problem for some later work, let us repeat the basic result of the second-order approximation of the outer flow. On the free surface of a steady stream over a horizontal plate there exists, in the case of subcritical flow, a stationary wave of small amplitude that spreads far downstream. This wave tail on the free surface is induced entirely by the effect of viscosity.

5. The second-order approximation – boundary layer

By substitution of (3) into (1), and by the assumption $\Psi^{(1)} = \Psi^{(1)}(x,\eta)$, we get an equation for $\Psi^{(1)}$ that is valid on the upper side of the plate:

$$\Psi^{(1)}_{\eta\eta\eta} + f\Psi^{(1)}_{\eta\eta} - 2xf'\Psi^{(1)}_{x\eta} + 2xf''\Psi^{(1)}_{x} + f'\Psi^{(1)}_{\eta}$$
$$= -j[(2x)^{\frac{3}{2}} k_0 b_w(F,0)\cos(k_0 x + \tfrac{1}{4}\pi)] - \frac{2\beta x}{1-F} + \surd 2 \sum_{N=1}^{\infty} (4N-1)\frac{b_N(F,0)}{x^{2N-1}}, \quad (24)$$

where the right-hand side is obtained by matching with the first two approximations of the outer flow. The coefficients $b_w(F,0)$ and $b_N(F,0)$ (see (19) and (23)) are

$$b_w(F,0) = \frac{\beta\pi\surd 2}{k_0^{\frac{1}{2}}\left(1 - \dfrac{F}{1-F^2k_0^2}\right)}, \quad b_1(F,0) = \frac{\beta(F^2-3F+1)}{6\surd 2(1-F)^2}, \quad \ldots,$$

and the parameter j introduced in (24) has the following values: $j = 1$ for $F < 1$ (subcritical flow) and $j = 0$ for $F > 0$ (supercritical flow). The first two boundary conditions for (24), obtained from (1 *c*), are

$$\Psi^{(1)}_{\eta} = 0, \quad \Psi^{(1)}_{x} = 0 \quad \text{for} \quad \eta = 0. \tag{25}$$

The third condition follows by matching with the outer flow:

$$\Psi^{(1)} \rightarrow j(2x)^{\frac{1}{2}} b_w(F,0) \sin(k_0 x + \tfrac{1}{4}\pi) - \frac{2\beta x}{1-F} - \sqrt{2} \sum_{N=1}^{\infty} \frac{b_N(F,0)}{x^{2N-1}} \quad \text{for} \quad \eta \rightarrow \infty. \tag{26}$$

To solve (24) we follow the usual methods for the higher approximations of boundary-layer theory (Van Dyke 1975). We assume, in accordance with the outer boundary conditions (26), that $\Psi^{(1)}$ has the form

$$\Psi^{(1)}(x,\eta) = \frac{jb_w(F,0)}{k_0^{\frac{1}{2}}} [(k_0 x)^{\frac{1}{2}} W_0(\eta) + (k_0 x)^{\frac{3}{2}} W_1(\eta) - \tfrac{1}{2}(k_0 x)^{\frac{5}{2}} W_2(\eta) - \ldots]$$
$$+ \frac{2\beta x}{1-F} L_0(\eta) + \frac{\sqrt{2} b_1(F,0)}{x} L_1(\eta) + \frac{\sqrt{2} b_2(F,0)}{x^3} L_2(\eta) + \ldots, \tag{27}$$

so that the unknown functions $W_0(\eta)$, $W_1(\eta)\ldots$ and $L_0(\eta)$, $L_1(\eta)\ldots$ satisfy

$$W'_0, W'_1, W'_2, \ldots, L'_0, L'_1, \ldots \rightarrow 1 \quad \text{for} \quad \eta \rightarrow \infty.$$

By substituting (27) into (24) and equating terms of the same powers of x, we obtain two infinite systems of ordinary differential equations:

$$\left.\begin{aligned} &W'''_N + fW''_N - 2Nf'W'_N + (2N+1)f''W_N = -2N, \\ &L'''_N + fL''_N + (4N-1)f'W'_N - 2(2N-1)f''W_N = 4N-1, \end{aligned}\right\} \quad N = 0,1,2,3,\ldots. \tag{28}$$

In the same way, from (25) and (26), we obtain the appropriate boundary conditions:

$$\left.\begin{aligned} &W_N = W'_N = 0, \quad L_N = L'_N = 0 \quad \text{for} \quad \eta = 0, \\ &W'_N \rightarrow 1, \quad L'_N \rightarrow 1 \quad \text{for} \quad \eta \rightarrow \infty, \end{aligned}\right\} \quad N = 0,1,2,3,\ldots.$$

From this procedure it is obvious that the functions $W_N(\eta)$ represent the wave disturbance of a Blasius boundary layer, the function $L_0(\eta)$ the parabolic disturbance, and the functions $L_1(\eta)$, $L_2(\eta)\ldots$ the local disturbance near the leading edge.

Equations (28) are independent, so they are easy to solve numerically. It was sufficient for our purposes to solve the first twenty-one W_N functions, and only the first four L_N functions, but there would be no difficulty in going further.

So, the second-order approximation of the horizontal velocity in the boundary layer (on the upper side of the plate) is

$$U^{(1)} = \Psi^{(1)}_Y = \frac{jb_w(F,0)}{\sqrt{2}} [W'_0 + (k_0 x) W'_1 - \tfrac{1}{2}(k_0 x)^2 W'_2 - \ldots]$$
$$+ \frac{\beta(2x)^{\frac{1}{2}}}{1-F} L'_0 + \frac{b_1(F,0)}{x^{\frac{3}{2}}} L'_1 + \frac{b_2(F,0)}{x^{\frac{7}{2}}} L'_2 + \ldots.$$

Two examples of this velocity component, one for subcritical and one for supercritical flow ($F = 0.75$ and $F = 1.5$), are presented in figure 3. Examples of the total horizontal

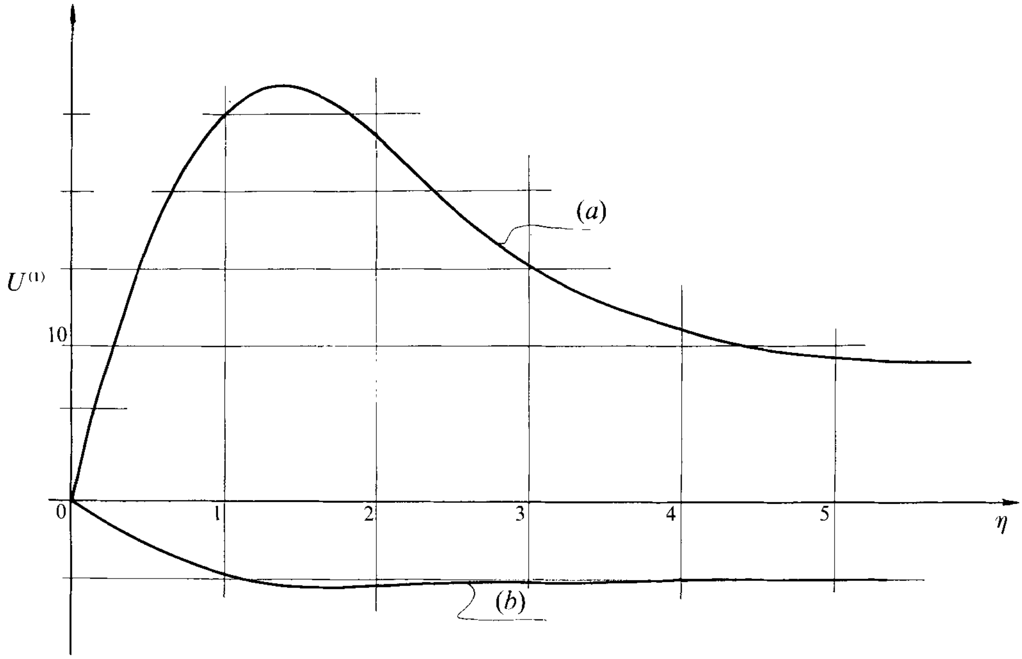

FIGURE 3. Second-order correction for the streamwise velocity in the boundary layer for $x = 2.5$: (a) subcritical flow ($F = 0.75$); (b) supercritical flow ($F = 1.5$).

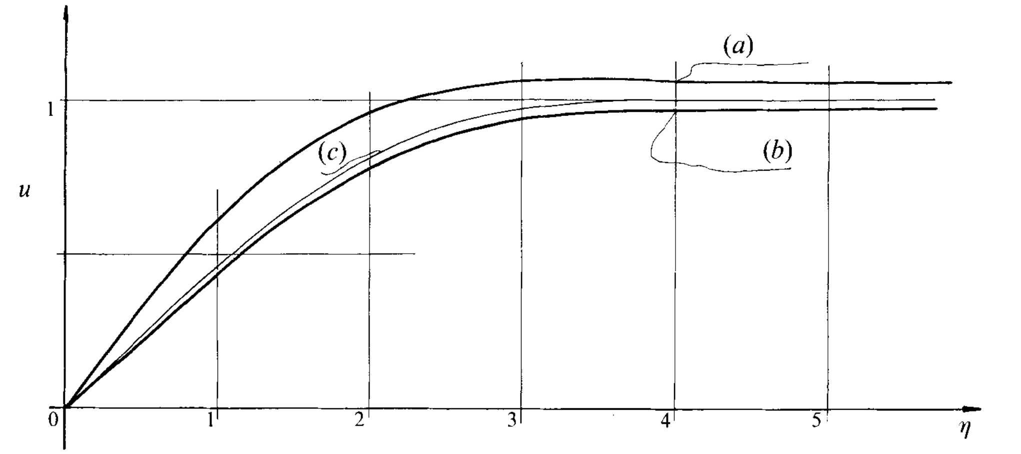

FIGURE 4. Total streamwise velocity in the boundary layer for $x = 2.5$, $1/R^{\frac{1}{2}} = 0.006$: (a) subcritical flow ($F = 0.75$); (b) supercritical flow ($F = 1.5$); (c) without the correction.

velocity (the sum of the first- and the second-order approximations), for the same values of F, are presented in figure 4. We note that, for the given liquid, F and R define the exact values of h and u_0 according to

$$h = \left(\frac{\nu^2}{g}\frac{R^2}{F}\right)^{\frac{1}{3}}, \quad u_0 = (g\nu FR)^{\frac{1}{3}}. \tag{29}$$

So, to get a better feel for the results, we also give the dimensional values for the last

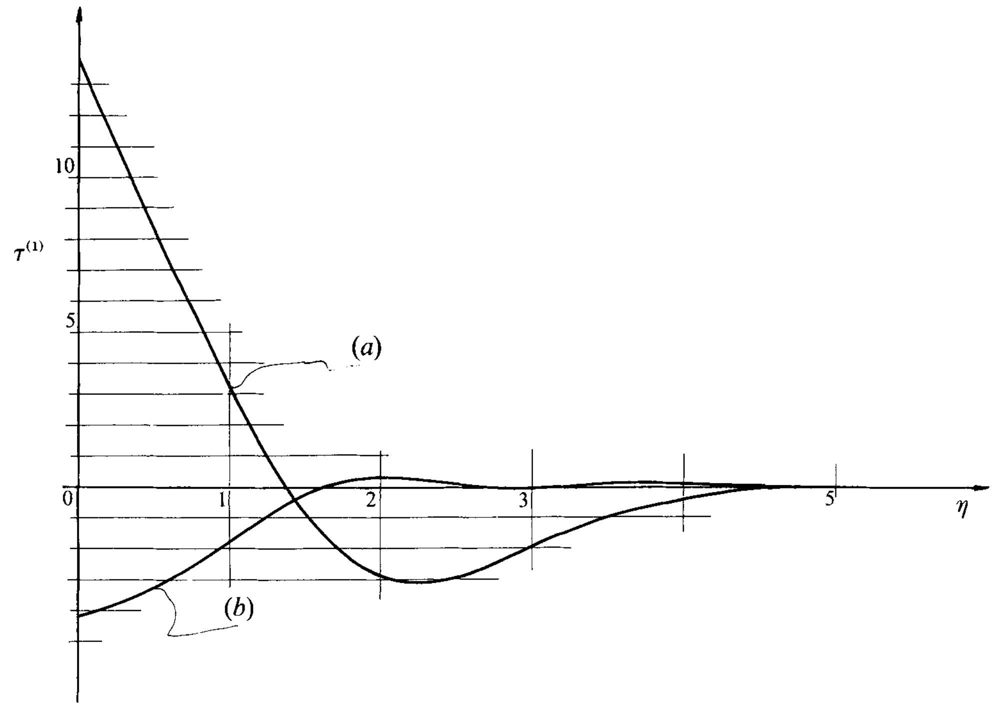

FIGURE 5. Second-order correction for the shear stress in the boundary layer for $x = 2.5$: (a) subcritical flow ($F = 0.75$); (b) supercritical flow ($F = 1.5$).

example. It corresponds to the plate travelling with a velocity of 0.589 m/s at 4.7 cm under the water surface for the subcritical flow, and with a velocity of 0.742 m/s at 3.7 cm under the water surface for the supercritical flow.

The second-order approximation of the shear stress in the boundary layer (on the upper side of the plate) is

$$\tau^{(1)} = F\Psi^{(1)}_{YY} = \frac{jFb_w(F,0)}{2x^{\frac{1}{2}}}[W''_0 + (k_0 x)\, W''_1 - \tfrac{1}{2}(k_0 x)^2 W''_2 - \ldots]$$

$$+\frac{\beta F}{1-F}L''_0 + \frac{F}{\sqrt{2}}\left[\frac{b_1(F,0)}{x^2}L''_1 + \frac{b_2(F,0)}{x^4}L''_2 + \ldots\right]. \quad (30)$$

Examples of $\tau^{(1)}$, for subcritical and supercritical flows (again for the values $F = 0.75$ and $F = 1.5$), are presented in figure 5. The total shear stress (the sum of the first two approximations) for the same values of F is presented in figure 6. One should note the characteristic increase of the stress near the plate in the case of subcritical flow. The example in figure 6 is given for a plate travelling with the same velocity at the same depth as the one in figure 4. The associated wave amplitude in these examples is extremely small from the point of view of water wave theory (just over a millimetre). Still, the increase of the stress near the plate is over 50%, exceeding, for that special case, even the perturbation method we used. It indicates the major effect that the boundary layer–free surface interaction has on the skin friction.

We also point out that, as should be expected from the previously obtained results for the outer flow, all three parts of $U^{(1)}$ and $\tau^{(1)}$ (wave, parabolic and local) grow unboundedly as the Froude number approaches its critical value ($F \to 1$).

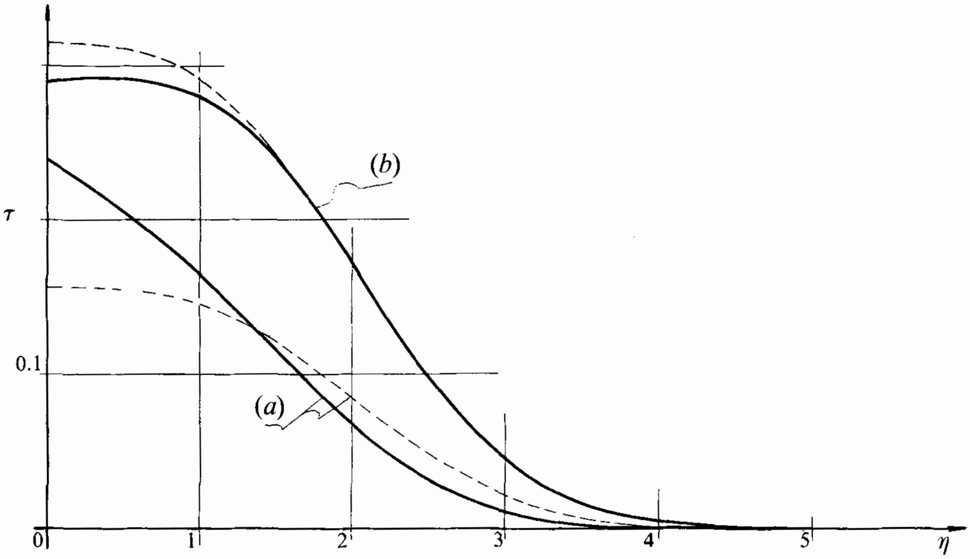

FIGURE 6. Total shear stress in the boundary layer for $x = 2.5$, $1/R^{\frac{1}{2}} = 0.006$: ($a$) subcritical flow ($F = 0.75$), ($b$) supercritical flow ($F = 1.5$); with the second-order correction (——), without the correction (------).

N	$W''_N(0)$	$L''_N(0)$	N	$W''_N(0)$
0	0.7044	1.9157	11	13.8704
1	2.8865	−2.3463	12	14.6973
2	4.4934	−8.121	13	15.5015
3	5.8605	−17.55	14	16.2855
4	7.0859		15	17.0510
5	8.2143		16	17.7997
6	9.2704		17	18.5331
7	10.2697		18	19.2521
8	11.2227		19	19.9581
9	12.1370		20	20.6517
10	13.0181			

TABLE 1. Numerical values of some variables in equation (31).

On the lower side of the plate, in accordance with the second-order approximation of the outer flow (13), it can be easily proved (in the same way as in the classical flow with no free surface) that we have:

$$U^{(1)} = 0, \quad \tau^{(1)} = 0.$$

The boundary layer under the plate is not influenced by the presence of the free surface.

5.1. *Local and integrated skin friction*

The second-order approximation of the coefficient of local skin friction for the upper side of the plate follows from (30):

$$c_f^{(1)}(x) = \tau^{(1)}(x,0) = \frac{jFb_w(F,0)}{2x^{\frac{1}{2}}}[W_0''(0)+(k_0 x)\,W_1''(0)-\ldots]$$

$$+\frac{\beta F}{1-F}L_0''(0)+\frac{F}{\sqrt{2}}\left[\frac{b_1(F,0)}{x^2}L_1''(0)+\ldots\right], \quad (31)$$

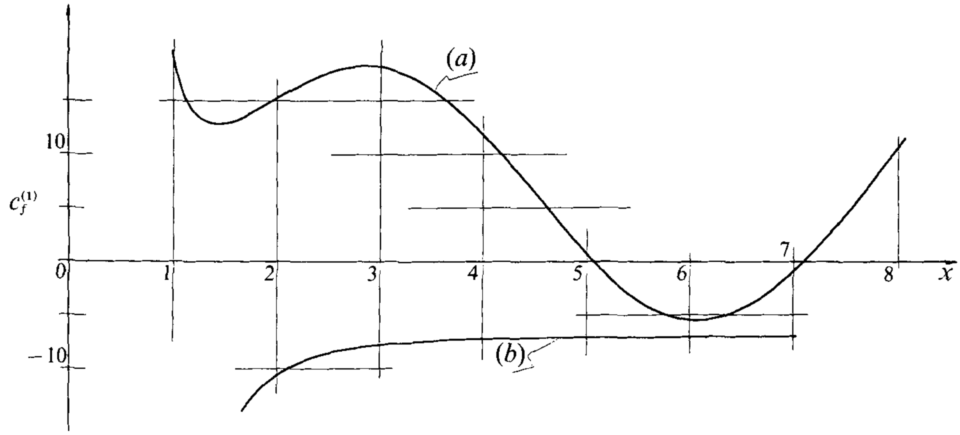

FIGURE 7. Second-order correction for the coefficient of local skin friction: (*a*) subcritical flow ($F = 0.75$); (*b*) supercritical flow ($F = 1.5$).

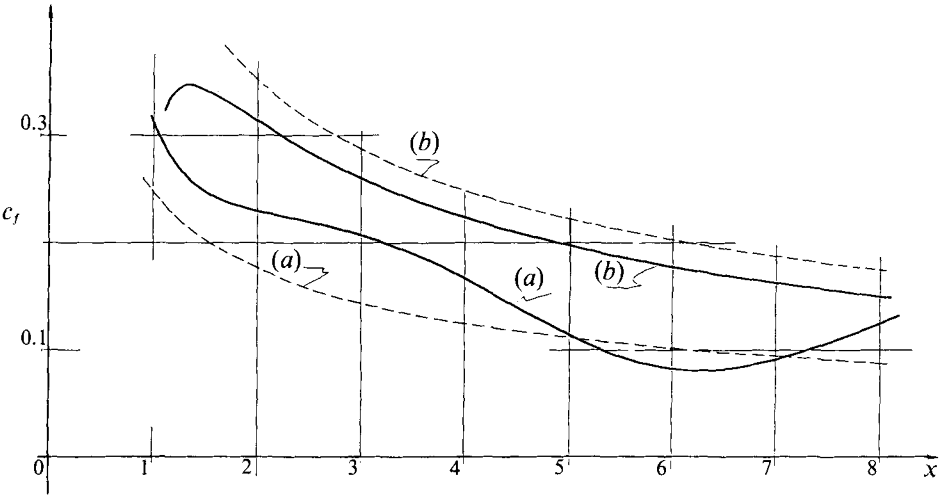

FIGURE 8. Coefficient of local skin friction for $1/R^{\frac{1}{2}} = 0.0035$: (*a*) subcritical flow ($F = 0.75$), (*b*) supercritical flow ($F = 1.5$); with the second-order correction (——), without the correction (------).

where we can still clearly recognize the parts due to the wave, parabolic and local displacement of the free surface. Some numerical values for $W''_N(0)$ and $L''_N(0)$ are given in table 1, and examples of function (31), for subcritical and the supercritical flow, are presented in figure 7. The sum of this second-order correction and the first (Blasius) approximation is presented in figure 8. We used a higher value of R in the last example than that in figures 4 and 6. According to (29), this change corresponds to placing the plate deeper under the surface, which is to avoid a great increase in skin friction (see figure 6), so as to make sure we stay well in the domain of validity of the theory used.

The second-order approximation of the coefficient of integrated skin friction:

$$c_F^{(1)} = \frac{1}{x}\int_0^x c_f^{(1)}(x)\,\mathrm{d}x$$

cannot be found by integration of (31), because of the unknown flow near the leading edge of the plate. The most we can do from (31) is to integrate the parts of $c_f^{(1)}$ due to

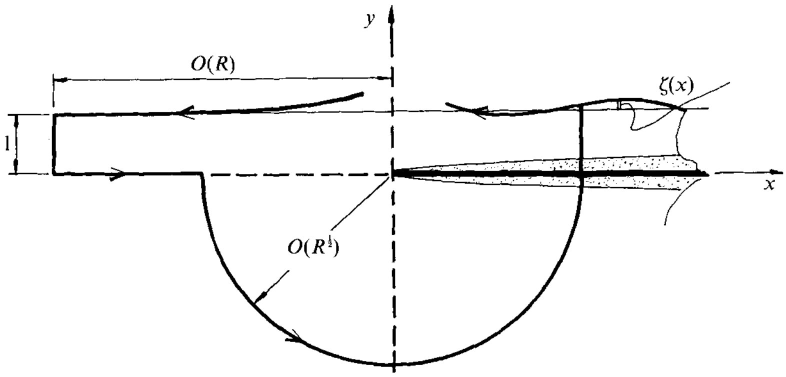

FIGURE 9. Contour considered for the balance of momentum.

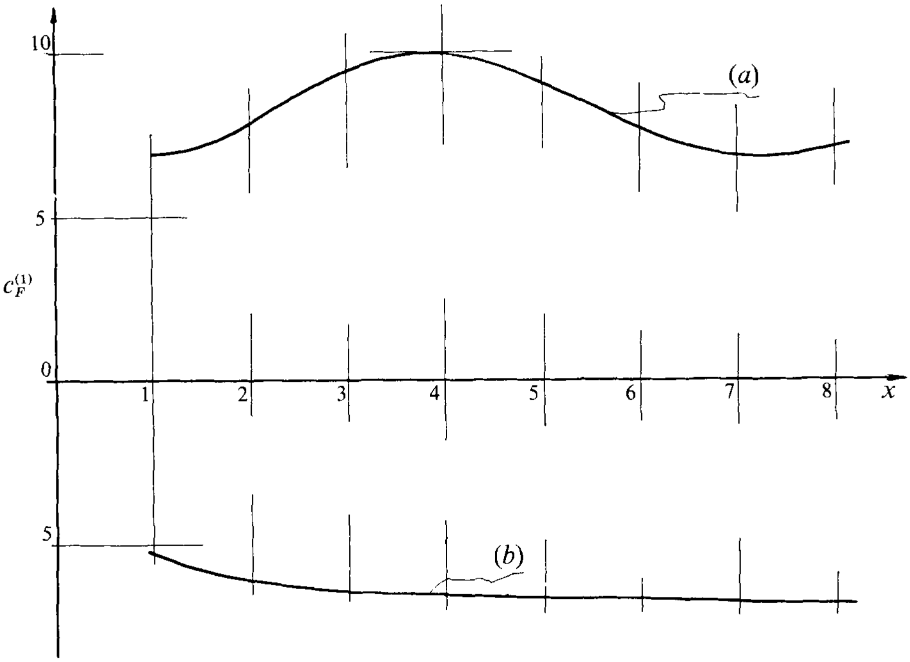

FIGURE 10. Second-order correction for the coefficient of integrated skin friction: (a) subcritical flow ($F = 0.75$); (b) supercritical flow ($F = 1.5$).

the wave and the parabolic disturbance. So, taking for convenience both sides of the plate (upper plus lower), we find

$$c_F^{(1)}(x) = \frac{\beta F L_0''(0)}{1-F} + \frac{jFb_w(F,0)}{x^{\frac{1}{2}}}\left[W_0''(0) + \tfrac{1}{3}(k_0 x)\, W_1''(0) - \tfrac{1}{10}(k_0 x)^2 W_2''(0) - \ldots\right] + \frac{D}{x}, \tag{32}$$

where D is an unknown friction force (independence of x) due to the local part of $c_f^{(1)}$ (also containing the possible influence of the lower side of the plate) – it is the leading-edge drag.

To avoid the difficulty associated with the unknown flow for small x, the classical idea (Imai 1957) of balance of the momentum in a large contour containing the leading edge is used. The contour presented in figure 9 was chosen, and the solutions for the outer flow and the boundary layer obtained earlier were used. The flow on the contour

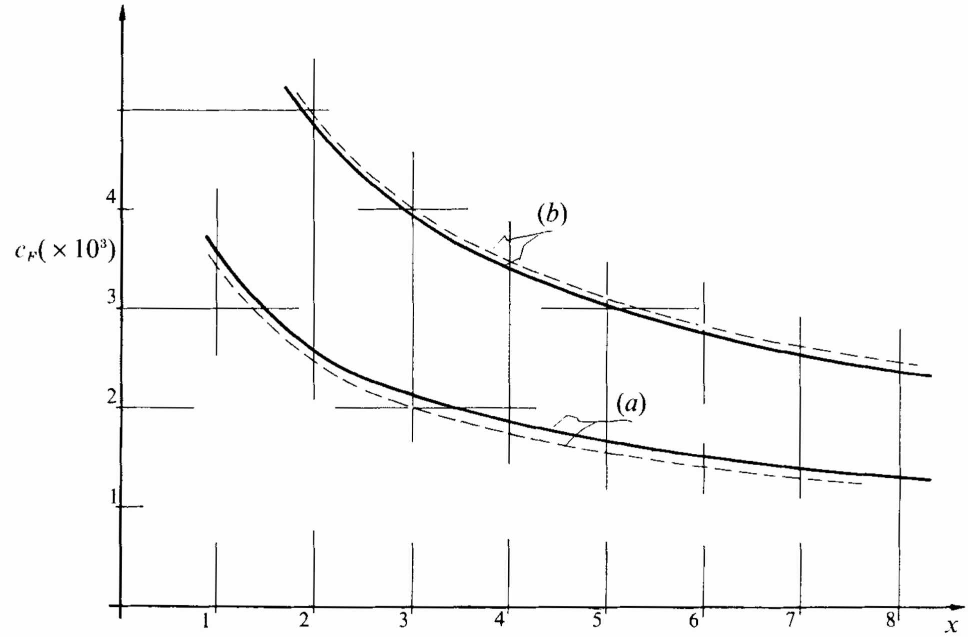

FIGURE 11. Coefficient of integrated skin friction for $1/R^{\frac{1}{2}} = 0.0035$: (*a*) subcritical flow ($F = 0.75$), (*b*) supercritical flow ($F = 1.5$); with the second-order correction (——), without the correction (------).

is known everywhere except at the free surface above the leading edge, but (as there is no flow through the free surface) this causes no difficulties in finding the total resistance of the plate. From the balance of momentum, after some calculation, the result that confirms (32) follows, provided that

$$D = \tfrac{1}{4}F\beta^2\pi + \frac{F\beta^2\pi}{2k_0(F\cosh^2 k_0 - 1)} \tag{33}$$

for subcritical flow, and

$$D = \tfrac{1}{4}F\beta^2\pi \tag{34}$$

for supercritical flow. The solution (34) may seem surprising, knowing that in the unbounded stream we would have

$$D = \tfrac{1}{2}F\beta^2\pi,$$

and it suggests that the leading-edge drag on the upper side of the plate disappears for supercritical flow.

The coefficient $c_F^{(1)}(x)$, from (32), (33) and (34), for subcritical flow ($F = 0.75$), and supercritical flow ($F = 1.5$) is presented in figure 10. By addition of this last result to the first-order approximation, the total coefficient of the integrated skin friction is found:

$$c_F(x) = \frac{1}{R^{\frac{1}{2}}}c_F^{(0)} + \frac{1}{R}c_F^{(1)} = \frac{4F\alpha}{(2Rx)^{\frac{1}{2}}} + \frac{\beta^2\pi F}{4Rx} + \frac{\beta F L_0''(0)}{R(1-F)}$$

$$+ j\frac{F}{R}\left\{\frac{\beta^2\pi}{2k_0 x(F\cosh^2 k_0 - 1)} - \frac{\beta(2\pi)^{\frac{1}{2}}}{(k_0 x)^{\frac{1}{2}}(F\cosh^2 k - 1)}[W_0''(0) + \tfrac{1}{3}(k_0 x)\, W_1''(0) + \ldots]\right\}$$

and is presented in figure 11. Finally, to stress the influence of the free surface, a different coefficient of skin friction $c_{FN} = c_F/F$ is introduced (this corresponds to a

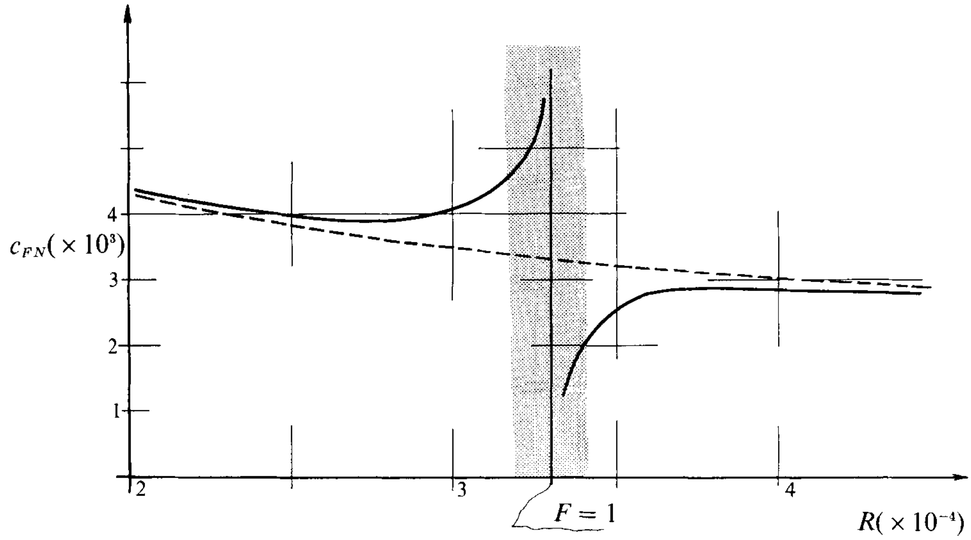

FIGURE. 12. Coefficient of integrated skin friction as a function of Reynolds number (for water, $x = 5$, $h = 0.05$ m): with the second-order correction (——), without the correction (------).

change of scale for pressure from ρgh to ρu_0^2, and that coefficient, as a function of Reynolds number for given x, is presented in figure 12. Note that this type of presentation (the c_{FN}–R diagram) is the most frequently used in practical calculations of plate resistance. The main results of the present theory, compared with the results that would be obtained for a deeply submerged plate, can be clearly recognized now. They are (see figure 12) a considerable increase of the resistance in the case of subcritical flow, a decrease of resistance in the case of supercritical flow, and an abrupt disturbance of flow in the transcritical region. The large changes of the resistance as the Froude number approaches its critical value cannot be treated by the present theory. Still, it gives a useful warning of the flow region where the classical ideas of independent analysis of the frictional and wave resistance break down.

5.2. *Lift*

The presence of the free surface causes a pressure difference on the two sides of the plate:

$$\Delta p = p'(x, 0^-) - p'(x, 0^+) = c_u^{(1)}(x)/R^{\frac{1}{2}},$$

where the local lift coefficient $c_u^{(1)}$ was introduced. From (4b), (7b) and from the results obtained for the outer flow (21), (22), it then follows that

$$c_u^{(1)}(x) = jb_w(F, 0)\sin(k_0 x + \tfrac{1}{4}\pi) + \frac{\beta(2x)^{\frac{1}{2}}}{1-F} + \sum_{N=1}^{\infty} \frac{b_N(F, 0)}{x^{2N-\frac{1}{2}}}.$$

Two examples of the change of lift coefficient along the plate, one for subcritical flow ($F = 0.75$) and one for supercritical flow ($F = 1.5$), are presented in figure 13. It is interesting to note that the lift obtained has different directions for the cases of subcritical and supercritical flow. Although the phenomenon that the horizontal plate below the free surface could produce a lift force is known (see Vanden-Broeck & Dias 1991), here we have a new situation. In contrast to the inviscid flow of Vanden-Broeck & Dias, the symmetry of the analysed flow is disturbed and consequently the lift force produced entirely by the effects of viscosity.

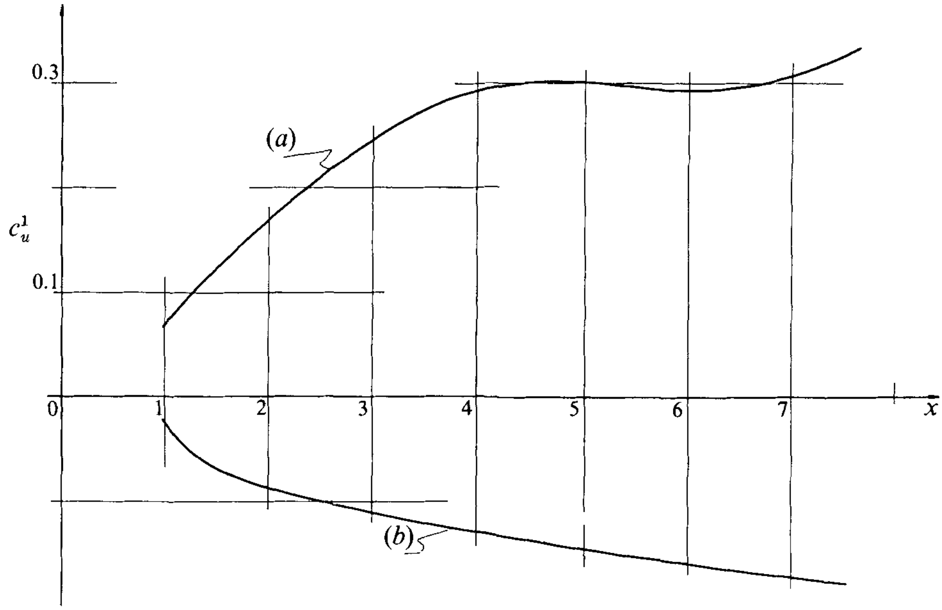

FIGURE. 13. Coefficient of local lift for $1/R^{\frac{1}{2}} = 0.006$: (*a*) subcritical flow ($F = 0.75$); (*b*) supercritical flow ($F = 1.5$).

6. Conclusion

By solving the problem of flow along a semi-infinite horizontal plate under a free surface by the method of matched asymptotic expansions, as the first-order approximation we obtained a flow with no free surface–boundary layer interaction. This result is consistent with Froude's method for determining the resistance by superimposing the effects of friction and waves, and it shows this classical approach in a new light, as the first term of the singular perturbation method used. It could be understood as a theoretical proof (at least for the problem studied here) of Froude's intuitive idea. It also gives the error incorporated in this classical method. It is of the order of omitted terms, that is $O(1/R^{\frac{1}{2}})$.

The second-order approximation gives the free surface–boundary layer interaction. In the outer flow, free-surface disturbances caused entirely by the effects of the viscosity were obtained. These disturbances have different forms for subcritical and supercritical regimes of flow. For the subcritical regime only, the effect of viscosity induces a gravity wave of small amplitude that spreads far downstream. The amplitude of this wave increases as the Froude number rises to its critical value, and there (in the transcritical flow regime) the presence of only a slight viscosity causes abrupt changes in the flow field.

The presence of the free surface, on the other hand, changes the flow in the boundary layer (on the upper side of the plate only), increasing the resistance in the case of subcritical flow, and decreasing it in the case of supercritical flow, all compared with the values for the unbounded stream. And that seems to be the most significant effect of boundary layer–free surface interaction. In the transcritical regime, as in the outer flow, a slight change of viscosity produces large changes in the boundary-layer flow. There, the classical methods cannot give even an approximate solution, so there Froude's old idea of the independent analysis of waves and friction breaks down.

Finally, the present theory predicts a vertical force (the lift) that would, as a result of the disturbed symmetry of the flow, act on the submerged plate.

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